

Precalculus Notes

1: Functions

1. A **function** is a "rule" that assigns every element x of 1 set – called Domain, D - to *exactly one* element, y , of another – called Range, R .

2. A function can be represented:

- a) Algebraically: $y = f(x)$
- b) Graphically
- c) via a Table of Values

3. The graph of a function always conforms to the **Vertical Line Test**: *any* vertical line intersects the graph in at most one point. Alternately, no two x s share the same y .

4. A function is **increasing** on an interval I if $f(a) < f(b)$ whenever $a < b$.

Likewise, a function is **decreasing** on an interval I if $f(a) > f(b)$ whenever $a > b$.

5. Types of basic functions:

a) Polynomial function: $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$

b) Rational Function: $f(x) = p(x)/q(x)$

c) Exponential Function: $f(x) = a^{f(x)}$

d) Logarithmic Function: $f(x) = \log f(x)$

e) Root function: $f(x) = \sqrt[n]{p(x)}$

6. **Domain** of a function, $y = f(x)$ is the set of *all* values of x for which $f(x)$ is "defined" ie. it exists / possesses a value. Therefore, it is the set of all value x can take! Sometimes, it is easier to specify the values that x *cannot* take.

Tip!

a) In general, the domain of a polynomial function, $y = f(x)$ is *all real numbers*.

b) For a rational function, $f(x) = p(x)/q(x)$, make sure that the denominator, $q(x)$, does *not* collapse to zero! Alternately, determine the values of x for which $q(x) = 0$; then, $f(x)$ exists for all *other* values.

c) For a function, $f(x) = \sqrt[n]{p(x)}$, (n is *even*), make sure that $p(x)$ inside the even-powered root, is *non-negative* ie. ≥ 0 .

d) For a function, $f(x) = \sqrt[n]{q(x)}$, (n is *odd*), the domain is *all real numbers*, since odd-powered roots of even negative numbers exist.

e) The domain is easily determined through the graph of the function: what are the values that x can / cannot take?

7. The **Range** of a function is the set of all values y can take.

Tip!

a) The Range is best determined *graphically*.

b) Determine the inverse function $f^{-1}(x)$. Then the domain of $f^{-1}(x)$ is the range of $f(x)$!

8. **Transformations:** examine **NOTES #2: Family of Functions!** Below.

a. **Vertical Shift** of a function: $y = f(x) + c$, $c > 0$, shifts the graph of $f(x)$ *up* c units. The *entire* graph gets "pulled up"!

Similarly, $y = f(x) - c$, $c > 0$, shifts (or "pushes") the graph of $f(x)$ *down* c units. The *entire* graph gets "pushed down"!

b. **Horizontal Shift** of a function: $y = f(x + c)$, $c > 0$, shifts the graph of $f(x)$ *to the left*, c units. The *entire* graph moves left (it is the "opposite" of what you'd think!).

Similarly, $y = f(x - c)$, $c > 0$, shifts the graph of $f(x)$ *to the right*, c units. The *entire* graph gets moves right (it is the "opposite" of what you'd think!).

c. **Vertical Stretch** of a function $y = c f(x)$, $c > 1$, *stretches* the graph vertically by a factor of c . When c is greater than 1, the *entire* graph gets "compressed" or "thinner"!

Similarly, $y = c f(x)$, $0 < c < 1$, *compresses* the graph vertically by a factor of c . When c is a positive fraction less than 1, the *entire* graph gets "flatter" or "fatter"!

d. **Reflection** of $f(x)$ about the axes: $y = -f(x)$ flips $f(x)$ about the **x-axis**.

Similarly, $y = f(-x)$ flips $f(x)$ about the **y-axis**.

9 a) A function $f(x)$ is an **even function** if $f(x) = f(-x)$ for all x in the domain of f . The graph of an even function is symmetrical with respect to the **y-axis**.

b) A function $f(x)$ is an **odd function** if $f(-x) = -f(x)$ for all x in the domain of f . The graph of an odd function is said to be symmetrical with respect to the origin.

10. Operations on functions:

$$f(x) + g(x) = (f + g)(x)$$

$$f(x) \cdot g(x) = (f \cdot g)(x)$$

$$f(x) - g(x) = (f - g)(x)$$

$$f(x)/g(x) = (f/g)(x), g(x) \neq 0$$

11. A **Piecewise** function is a function whose definition involves more than one formula. These graphs may have sharp corners.

12. The graph of a function is **continuous** if it has no "holes" or "breaks" and **discontinuous** at any point where it has a hole or break.

13 a) If z varies **directly** (or **jointly**) with x and y then $z = kx \cdot y$ where k is the constant of proportionality.

b) If z varies **inversely** with x and y then $z = k/(x \cdot y)$ where k is the constant of proportionality.

2: Inequalities

1. **Polynomial Functions:** To solve an inequality of the form

$$f(x) > 0 \text{ or}$$

$$f(x) < 0 \text{ or}$$

$$f(x) \geq 0 \text{ or}$$

$$f(x) < 0,$$

where $f(x)$ is a Polynomial Function of the form, $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$
[NOTE: if there are terms on both sides of the inequality, transpose all the terms to one side *first* to get $f(x)$!]

- a) Express $f(x)$ as a product of linear or *irreducible* quadratic function [a quadratic function that cannot be factored into linear terms because of *complex* roots]
- b) Determine the roots / zeros of each linear term constituting $f(x)$.
- c) Keeping in mind that an irreducible quadratic function is *always* positive or *always* negative, choose any “test value” to determine which is the case.
- d) Mark the roots of $f(x)$ on a number line.
- e) Choose a “test value” for x in each interval, and determine the sign of $f(x)$ for that test value by substitution, keeping c) in mind [if applicable].

Remember: we are NOT interested in the actual *value* of the function for the test points in each intervals, *only* the *sign* of $f(x)$.

Tip!

If $f(x)$ is composed of *only* linear terms, then a test value for only one interval [any interval!] need be evaluated to determine the sign of $f(x)$ for that value. For the *other* intervals, the sign of $f(x)$ simply *alternates*!

- f) Use the signs of $f(x)$ to state the solution. Note the type of inequality [$>$ or \geq , $<$ or \leq] to decide whether to *include* or *exclude* the *boundaries* of each interval!

2. Rational Functions: If $f(x)$ is a Rational Function of the form, $f(x) = p(x)/q(x)$, then

- a) Express $p(x)$ and $q(x)$ as a product of linear or *irreducible* quadratic function [a quadratic function that cannot be factored into linear terms because of *complex* roots]
- b) Determine the roots / zeros of each linear term constituting $p(x)$ and $q(x)$.
- c) Keeping in mind that an irreducible quadratic function is *always* positive or *always* negative, choose any “test value” to determine which is the case.
- d) Mark the roots of $p(x)$ and $q(x)$ on a number line.
- e) Choose a “test value” for x in each interval, and determine the sign of $f(x)$ for that test value by substitution, keeping c) in mind [if applicable].

Remember: we are NOT interested in the actual *value* of the function for the test points in each intervals, *only* the *sign* of $f(x)$.

Tip!

If $f(x)$ is composed of *only* linear terms, then a test value for only one interval [any interval!] need be evaluated to determine the sign of $f(x)$ for that value. For the *other* intervals, the sign of $f(x)$ simply *alternates*!

- f) Use the signs of $f(x)$ to state the solution. Note the type of inequality [$>$ or \geq , $<$ or \leq] and the *domain* of $f(x)$ [that is, exclude values that make $q(x)$ ZERO!] to decide whether to *include* or *exclude* the *boundaries* of each interval!

3: Quadratic Functions & Equations

1. The function, $f(x) = ax^2 + bx + c$ is a quadratic function (QF), whose graph is a parabola. And $f(x) = ax^2 + bx + c = 0$ is the associated quadratic equation (QE).

2. The **roots** or **zeros** or **solutions** of a QE give the x-intercepts of the corresponding QF, where the parabola cuts the x-axis.

3. The 2 roots of the QE: $f(x) = ax^2 + bx + c = 0$ are $x = -b \pm \sqrt{(b^2 - 4ac)} / 2a$. The **Sum of the roots** of a QE is *always* $-b/2a$, and the **Product of the roots** is *always* c/a .

4. $D = \sqrt{(b^2 - 4ac)} / 2a$ is called the **Discriminant** of the QE.

a) If $D > 0$, then the QE has 2 real, distinct roots, and the related QF / parabola has 2 x-intercepts.

b) If $D = 0$, then the QE has 1 real, repeat root, and the related QF / parabola has 1 x-intercept (ie. It only *touches* the x-axis, but does *not* cross it!)

c) Finally, if $D < 0$, then the QE has 2 imaginary, conjugate roots of the form: $a \pm bi$; and the related QF / parabola has *no* x-intercepts.

5. $f(x) = ax^2 + bx + c$ is the **Standard Form** of the parabola. Its **Axis of Symmetry (AOS)** is the *vertical* line $x = -b/2a$ that acts like the "mirror", about which the parabola is symmetrical.

The **Vertex** of the parabola is its highest / lowest point, and it is given by $[-b/2a, f(-b/2a)]$ ie. The x-coordinate is the AOS, while the y-coordinate is obtained by "plugging in" the AOS into $f(x)$.

If the leading coefficient, $a > 0$, then the parabola opens up, else down!

The **y-intercept** of the parabola is where it crosses the y-axis: $x = 0 \Rightarrow y = c$.

The **x-intercepts** are obtained solving the corresponding QE: $f(x) = ax^2 + bx + c = 0$.

Tip!

The **Axis of Symmetry** of the parabola is exactly in-between its 2 x-intercepts!

6. $f(x) = a(x - h)^2 + k$ is the **Vertex Form**, obtained by "completing the square" of the standard form: $f(x) = ax^2 + bx + c = a(x + b/2a)^2 + (c - b^2/4a)$ (See 6. above).

Its **Axis of Symmetry** is $x = h$, and its **Vertex** is (h, k) .

If the leading coefficient, $a > 0$, then the parabola opens up, else down!

The **y-intercept** of the parabola is obtained by "plugging in" $x = 0$.

The **x-intercepts** are obtained solving the corresponding QE: $f(x) = a(x - h)^2 + k = 0$.

7. $f(x) = a(x - p)(x - q)$ is the **Intercept Form** of the parabola, obtained by "factoring" of the standard form.

Its **Axis of Symmetry (AOS)** is $x = (p + q)/2$ since the AOS is exactly in-between its 2 x-intercepts! Its **Vertex** is obtained by "plugging in" the AOS into $f(x)$ for x .

If the leading coefficient, $a > 0$, then the parabola opens up, else down!

The **y-intercept** of the parabola is obtained by "plugging in" $x = 0$: $y = a.p.q$

The **x-intercepts** are, of course, p and q .

4: Composite Functions & Inverse Functions

1. A **Composite Function** is of the form $h(x) = f(g(x))$ so that, for some value k , $h(k) = f(g(k))$.

The domain of $h(x)$ is obtained by inspecting, *not only* the function $h(x)$ directly, *but also* the domain of $g(x)$ since, depending on *its* domain, $h(x)$ may not assume certain values! In other words, “merge” the domains of $g(x)$ and $f(g(x))$ to obtain the domain the $h(x)$.

2. A function with **Domain, D** and **Range, R**, is called **One-to-One** if either of the following is true:

a) If $a < b$ in **D**, then $f(a) < f(b)$ in **R**, or

b) If $f(a) = f(b)$ in **R**, then $a = b$ in **D**.

Tip!

If the *same* y -value can be obtained by more than one x -value, then $f(x)$ is *not* one-to-one!

3. **Horizontal Line Test:** A function $f(x)$ is **one-to-one** if and only if any horizontal line intersects its graph in at most 1 point.

Tip!

Increasing and *decreasing* functions are one-to-one functions (since a horizontal line cuts them at one point only!).

4. **Inverse Functions & Their Properties:** If $y = f(x)$ is a *one-to-one* function with Domain, D , and Range, R , then its **Inverse**, denoted as $f^{-1}(x)$ [and read as *f-inverse-of-x*], with Domain, R , and Range, D , is such that, for every x in D and every y in R ,

a) $y = f(x)$ if and only if $x = f^{-1}(y)$. That is, if (a, b) lies on $f(x)$, then (b, a) lies on $f^{-1}(x)$.

b) Domain of $f(x) \sim$ Range of $g(x)$; Range of $f(x) \sim$ Domain of $g(x)$

c) $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

d) To find the inverse function, $f^{-1}(x)$, *first*, insure that $f(x)$ is one-to-one via the Horizontal Line Test; *then* swap x and y in $y = f(x)$ and solve for y . Then, confirm that c) is True!

e) The graphs of $f(x)$ and $f^{-1}(x)$ are symmetrical about the 45° line $y = x$.

5: Exponential Functions

1. The function $y = a^x$ is the general exponential function, where a is a constant $\neq 0$ or 1 (for example, 2 , $-\frac{1}{2}$, 4.5 , 500 , -6 , -200 , $\frac{1}{4}$, etc).

Some basic facts:

- If $a > 1$, then $a^\infty = \infty$.
- If a is a positive number between 0 and 1 (a fraction like $\frac{1}{2}$ or $\frac{1}{4}$ or $\frac{3}{8}$), then $a^\infty =$ ZERO.
- $a^{-b} = 1 / a^b$
- If a is a positive number between 0 and 1 , and $b = 1/a$, then $b > 1$, and $a^{-x} = b^x$.

2. If $a > 1$, $y = a^x$ is called the **exponential growth function** and it's graph *always* behaves in the following manner:

- a. It rises from left to right slowly above the x-axis (which is the asymptote on the left) in the 2nd Quadrant, till it crosses (0, 1). To the right of the y-axis, the graph rises quickly!
- b. As $x \rightarrow -\infty$, (a large negative #), $y \rightarrow a^{-\infty} \sim 1/a^{\infty} \sim 1/\infty \sim \text{ZERO}$. In other words, the x-axis or $y = 0$ is the asymptote of the graph $y = a^x$.
- c. As $x \rightarrow \infty$ (a large positive #), $y \rightarrow a^{\infty} \sim \infty$ (a large positive #), also.
- d. It passes through (0, 1) since for $x = 0$, $y = a^0 = 1$.
- e. How fast the graph rises beyond (0, 1) depends on how large a is! So all $y = a^x$ graphs resemble each other; only a determines how steeply it shall rise.
- f. It's Domain is: $x \sim \text{ALL real numbers}$; its Range is: $y > 0$.

3. If a is a number such that $0 < a < 1$, then $y = a^x$ is the **exponential decay function**. If $b = 1/a$, then, $y = a^x$ is equivalent to $y = b^{-x}$ where $b > 1$. The graph of the exponential decay function *always* behaves in the following manner:

- a. It is the mirror image of $y = 1/a^x = b^x$ (where $b > 1$) about the y-axis.
- b. It falls quickly from left to right, in the 2nd Quadrant, till it crosses (0, 1). To the right of the y-axis – in the 1st Quadrant, the graph falls gently approaching the x-axis!
- c. As $x \rightarrow -\infty$, (a large negative #), $y \rightarrow a^{-\infty} \sim 1/a^{\infty}$ (remember: a is a fraction!) $\sim b^{\infty} \sim \infty$.
- d. As $x \rightarrow \infty$ (a large positive #), $y \rightarrow a^{\infty}$ (remember: a is a fraction!) $\sim 1/b^{\infty} \sim 1/\infty \sim \text{ZERO}$. In other words, the x-axis or $y = 0$ is the asymptote of the graph $y = a^x$.
- e. It passes through (0, 1) since for $x = 0$, $y = a^0 = 1$.
- f. How fast the graph falls before (0, 1) depends on how large a is! So all $y = a^x$ graphs resemble each other; only a determines how quickly it shall descend.
- g. It's Domain is: $x \sim \text{ALL real numbers}$; its Range is: $y > 0$

4. If an initial quantity, I_0 grows at a constant rate, r , for a time, t , then the final quantity, $I_t = I_0 e^{rt}$.

5. Graphing Exponential Functions of the General Form: $y = b \cdot a^{x+c} + d$, where a , b , c and d are constants.

- a. Graph the generic $y = a^x$ function passing through (0, 1) (See points 2. or 3. *above*, as may be applicable!)
- b. Graph the $y = a^{x+c}$ function by transforming ie. shifting the $y = a^x$ graph c units left / right (depending on the sign of c : if $c > 0$, shift to the left; if $c < 0$, then shift to the right). Now, the curve passes through $(-c, 1)$.
- c. Graph the $y = b \cdot a^{x+c}$ function by transforming ie. "pulling up" (if $b > 1$) or "flattening" to the side (if $0 < b < 1$), the $y = a^{x+c}$ graph by a factor of b . The graph of $y = b \cdot a^{x+c}$ now passes through $(-c, b)$ [substituting $x = -c$ into $y = b \cdot a^{x+c}$!].
- d. If $b < 0$, then flip graph about the x-axis.
- e. Finally, graph the $y = b \cdot a^{x+c} + d$ function by transforming ie. shifting the $y = b \cdot a^{x+c}$ curve up / down d units (depending on the sign of d : if $d > 0$, shift up; if $d < 0$, then shift down). The graph of $y = b \cdot a^{x+c} + d$ now passes through $(-c, b + d)$ [substituting $x = -c$ into $y = b \cdot a^{x+c} + d$!].

- f. The Domain of $y = b \cdot a^{x+c} + d$ is x : ALL real numbers; the Range is: $y > d$ or $y < d$.
- g. The asymptote is: $y = d$. [Examine the behavior of the graph as $x \rightarrow -\infty$, and $x \rightarrow \infty$.]
- h. For the sake of accuracy, calculate a table of values for x and y . For this, choose a couple of points for x , to the left and right of $x = -c$.

EXAMPLE: To graph: $y = -\frac{1}{2} 3^{x-4} + 1$

Tip! Draw each of the graphs in Steps a) – e) afresh on a new coordinate axis.

- a. Graph $y = 3^x$ function, rising left to right slightly above the x -axis (which is the asymptote) till it crosses $(0, 1)$ and then rising somewhat steeply to the right beyond the y -axis.
- b. Graph the $y = 3^{x-4}$ function by transforming ie. shifting the $y = 3^x$ graph 4 units right. Now, the curve passes through $(4, 1)$.
- c. Graph the $y = \frac{1}{2} \cdot 3^{x-4}$ function by transforming ie. "flattening" to the side the $y = 3^{x-4}$ graph by a factor of $\frac{1}{2}$. The graph of $y = \frac{1}{2} \cdot 3^{x-4}$ now passes through $(4, \frac{1}{2})$ [substituting $x = 0$ into $y = \frac{1}{2} \cdot 3^{x-4}$!].
- d. Since $b = -\frac{1}{2} < 0$, flip the graph of $y = \frac{1}{2} \cdot 3^{x-4}$ about the x -axis. So, the curve now passes through $(4, -\frac{1}{2})$.
- e. Finally, graph the $y = -\frac{1}{2} 3^{x-4} + 1$ function by transforming ie. shifting the $y = -\frac{1}{2} 3^{x-4}$ curve up 1 unit. The graph of $y = -\frac{1}{2} 3^{x-4} + 1$ now passes through $(4, \frac{1}{2})$ [substituting $x = 4$ into $y = -\frac{1}{2} 3^{x-4} + 1$!].
- f. The Domain of $y = -\frac{1}{2} 3^{x-4} + 1$ is x : ALL real numbers; the Range is: $y < 1$.
- i. The asymptote is: $y = 1$ since as $x \rightarrow \infty$, $y \rightarrow -\infty$, and as $x \rightarrow -\infty$, $y \rightarrow -\frac{1}{2} \cdot 3^{-\infty} + 1 \sim -\frac{1}{2} (1/3^\infty) + 1 \sim \text{ZERO} + 1 \sim \text{ONE}$.
- j. For the sake of accuracy, choose a few points for x , to the left and right of $x = 4$. For Y -scale, use: $(\text{Max} - \text{Min} / 10) \sim (1 - -13) / 10 = \sim 1.5$ or 2
- k.

x	2	3	4	5	6	7
y	$17/18 \sim 0.9$	$5/6 \sim 0.8$	0.5	-0.5	-3.5	-12.5

EXAMPLE: To graph: $y = 2 \cdot (\frac{1}{4})^{x+1} - 3$

Tip! Draw each of the graphs in Steps a) – e) afresh on a new coordinate axis.

- a. Graph $y = (\frac{1}{4})^x$ function, which falls quickly from left to right above the x -axis in the 2nd Quadrant, till it crosses $(0, 1)$. To the right of the y -axis in Quadrant 1, the graph falls gently, approaching the x -axis (which is the asymptote as $x \rightarrow \infty$).
- b. Graph the $y = (\frac{1}{4})^{x+1}$ function by shifting the $y = (\frac{1}{4})^x$ graph 1 unit left. Now, the curve passes through $(-1, 1)$.
- c. Graph the $y = 2 \cdot (\frac{1}{4})^{x+1}$ function by "stretching up" the $y = (\frac{1}{4})^{x+1}$ graph by a factor of 2. The graph of $y = 2 \cdot (\frac{1}{4})^{x+1}$ now passes through $(-1, 2)$ [substituting $x = -1$ into $y = 2 \cdot (\frac{1}{4})^{x+1}$!].
- d. Finally, graph the $y = 2 \cdot (\frac{1}{4})^{x+1} - 3$ function by shifting the $y = 2 \cdot (\frac{1}{4})^{x+1}$ curve down 3 units. The graph of $2 \cdot (\frac{1}{4})^{x+1} - 3$ now passes through $(-1, -1)$ [substituting $x = -1$ into $y = 2 \cdot (\frac{1}{4})^{x+1} - 3$!].
- e. The Domain of $y = 2 \cdot (\frac{1}{4})^{x+1} - 3$ is x : ALL real numbers; the Range is: $y > -3$.

- f. The asymptote is: $y = 3$ since as $x \rightarrow \infty, y \sim 2 \cdot (1/4)^\infty - 3 \sim 2 \cdot (1/4^\infty) - 3 \sim 2 \cdot \text{ZERO} - 3 \sim -3$, and as $x \rightarrow -\infty, y \sim 2 \cdot (1/4)^{-\infty} - 3 \sim 2 \cdot (4)^\infty - 3 \sim \infty$

For the sake of accuracy, choose a few points for x , to the left and right of $x = -1$.

6. The process for graphing exponential functions of the form $y = a \cdot e^{bx+c} + d$ or $y = a \cdot e^{-bx+c} + d$ is identical. Only remember: e is a constant $\sim 2.7128!$

6: Logarithmic Functions

1. Definition: If $a^b = c$, then $b = \log_a c$. Observe that when *logs* are taken for both sides of an exponential expression, then the # by itself (c) gets the *log*, with the base (a) of the exponent getting “pushed down” to become the base of the *log* function, and the exponent itself (b) is “pushed down”. Usually, the base a is 10 so that $\log x \sim \log_{10} x$.

Converse: If $\log_p q = r$, then $p^r = q$. Here the opposite occurs: we’re removing the *log* to go to exponential format, so the base (p) gets “pushed up” to become the base of the exponential expression and the # by itself (r) gets “pushed up” to become the exponent.

2. Natural *logarithms*: If the base for a *log* function is $e \sim 2.7128$ (a constant - like π - with “special” properties, used in advanced Math), then it is denoted as \ln (instead of *log*) so that $\ln x \sim \log_e x$.

Significance of e : As x becomes larger and larger, the expression $(x + 1/x)^x$ approaches $e \sim 2.7182$. [Demonstrate this by choosing $x = 5, 10, 100, 500, 25000$ and $500000!$]

3. Properties of *logarithms*:

- a) $\log_a a = 1$ since $a^1 = a$ [using the Converse].
- b) $\ln e = 1$ since $e^1 = e$.
- c) $\log_a 1 = 0$ since $a^0 = 1$ [using the Converse].
- d) $\log xy = \log x + \log y$ [$\neq (\log x) \cdot (\log y)$]
- e) $\log (x / y) = \log x - \log y$ [$\neq (\log x) / (\log y)$]
- f) $\log x^y = y \cdot \log x$
- g) $\log_a 0 = \text{Undefined}$ [since there’s no number c such that $a^c = 0$.]
- h) $\log_a b = \text{Undefined}$ if $b < 0$ [[since there’s no number c such that $a^c < 0$.]
- i) $\log_a b = \log_c b / \log_c a$ [Change of Base formula]

4. *Log* functions are the inverse functions of *exponential* functions. That is, the graphs of $f(x) = a^x$ and $g(x) = \log_a x$ satisfy the following properties:

- a. $f(g(x)) = g(f(x)) = x$
- b. If (a, b) lies on $f(x)$ then (b, a) lies on $g(x)$.
- c. The domain of $f(x)$ is the range of $g(x)$ and *vice versa*.
- d. The graphs of $f(x)$ and $g(x)$ are symmetric about the $y = x$ line.

5. The Domain of $f(x) = \log_a [g(x)]$ where $g(x)$ is a polynomial or rational function is: $g(x) > 0$ [because of **Property #3g** and **#3h** above]. The Range of $f(x) = \log_a [g(x)]$ is: All Real Numbers.

6. The graph of the **logarithmic function**, $y = f(x) = \log_a x$ always rises very swiftly from the 4th quadrant just to the right of the y -axis [since Domain is: $x > 0$], passes through (1, 0) and then rises gradually so that it is flatter beyond (1, 0) in the 1st quadrant.

7. The Domain and Range, and overall shape of the logarithmic function: $y = b \cdot \log_a (x + c) + d$, where a, b, c and d are constants, can be obtained easily by finding the inverse (exponential) function and applying #4 above.

8. Solving Exponential Equations:

a) Form $a^{f(x)} = b^{g(x)}$ where a is a power of b [$a = b^k$] or *vice versa* [$b = a^k$]: Make the bases identical by substituting $a = b^k$ or $b = a^k$, as the case may be. Then simplify, and use principle: if $x^y = x^z$ then $y = z$.

b) Form $a^{f(x)} = b$: convert to logarithm equation, $\log_a b = f(x)$ and use Property 3 (i) to simplify the left-hand-side.

9. **Solving Logarithmic Equations:** Use Properties of Logarithms [#3], first, if necessary to “isolate” the \log terms to one side.

a) Form $\log_a f(x) = b$: convert to exponential form, $f(x) = a^b$ and solve.

b) Form $\log f(x) = \log g(x)$: set $f(x) = g(x)$

7: Polynomial Functions & Equations

1. A function of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$ is called a **Polynomial Function of Degree n** . $a_n x^n$, the term with the highest power, is the **Leading Term** with a_n being the **Leading Coefficient**, and a_0 is the **Constant** term.

2. The graph of a polynomial function of degree n

a) Is continuous for all real numbers

b) Has *at most* $(n - 1)$ Turning Points.

c) Has no sharp corners

d) Has at most n real zeros.

e) Increases or decreases without bound as x approaches infinity and as x approaches negative infinity.

3. A polynomial function $f(x)$ can be divided by a linear expression $(x - a)$ by Long Division or by **Synthetic Division**.

4. **Division Algorithm for Polynomials:** If a polynomial, $f(x)$, of degree n , is divided by a linear expression $(x - a)$, then $f(x) = (x - a) \cdot q(x) + R(x)$ where $q(x)$ is a polynomial of degree $n - 1$ and $R(x)$ is the remainder.

5. **Remainder Theorem:** If a polynomial function $f(x)$ is divided by $(x - a)$, then the Remainder is $f(a)$. The remainder can be obtained by Direct Substitution [ie. $f(a)$] or by Synthetic Division.

6. **Factor Theorem:** If $(x - a)$ is a factor of a polynomial function $f(x)$, then the Remainder, $R = f(a) = 0$.

Converse: If, when a polynomial function, $f(x)$ is divided by $(x - a)$ the Remainder, $R = f(a) = 0$, then $(x - a)$ is a factor of $f(x)$.

7. The terms “roots”, “solutions”, “zeros” and “x-intercepts” are used interchangeably.

8. A polynomial function, $f(x)$, of degree n has exactly n roots [Real (rational + irrational) + Imaginary]. Some of these roots might be of different *multiplicities*, ie. repeat roots: if $f(x)$ is factored as a product of linear factors, the number of linear factors that have roots, r , is said to be the multiplicity of r .

9 a) Irrational Roots of the form $\mathbf{a + \sqrt{b}}$ always occur as Conjugates. That is, if $\mathbf{a + \sqrt{b}}$ is a root of a polynomial function, $f(x)$, then $\mathbf{a - \sqrt{b}}$ is also a root.

b) Complex Roots of the form $\mathbf{a + bi}$ always occur as Conjugates. That is, if $\mathbf{a + bi}$ is a root of a polynomial function, $f(x)$, then $\mathbf{a - bi}$ is also a root.

11. Every polynomial function, $f(x)$, can be expressed as a product of linear and irreducible quadratic factors ie. $f(x) = a(x - r_1)(x - r_2)...(x^2 + b_1x + c_1)(x^2 + b_2x + c_2)...$

12. **Rational Roots Theorem:** If a polynomial function $f(x) = a_nx^n + a_{n-1}x^{n-1} + ...a_2x^2 + a_1x + a_0$ of degree n (where a_n is the leading coefficient, and a_0 is the constant term) has integer coefficients $a_0, a_1, ...a_n$, then every rational root is of the form p/q where p is a factor of the constant term, a_0 , and q is a factor of the leading coefficient, a_n .

13 a) A number $a > 0$ is an **Upper Bound** for the real zeros of $f(x)$ if, when $f(x)$ is divided by $(x - a)$ by synthetic division, all numbers in the quotient row, including the remainder, are nonnegative.

b) A number $a < 0$ is a **Lower Bound** for the real zeros of $f(x)$ if, when $f(x)$ is divided by $(x - a)$ by synthetic division, all the numbers in the quotient row, including the remainder, alternate in sign.

c) **Mean Value Theorem:** If $f(a)$ and $f(b)$ have opposite signs, there exists an x-intercept between a and b .

15. If $f(x)$ has real coefficients, then

a) If a is a real zero of even multiplicity, then $f(x)$ has a turning point at a and does not change sign at a .

If a is a real zero of odd multiplicity, then $f(x)$ does not have a real turning point at a and changes sign at a .

16. **End Behavior** of a Polynomial Function:

a) As $x \rightarrow \infty$, $y \rightarrow +\infty$ or $-\infty$? In other words, if x is a large positive number, is y a large positive or negative number?

b) As $x \rightarrow -\infty$, $y \rightarrow +\infty$ or $-\infty$? In other words, if x is a large negative number, is y a large positive or negative number?

To determine the end behavior of a polynomial $f(x)$ of degree n , only consider the sign of the leading coefficient and the degree: the end-behavior of the graph of $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$ is similar to that of $g(x) = a_n x^n$. That is, as $x \rightarrow \infty$, $f(x) \approx g(x) = a_n x^n$.

16. Sketching polynomial functions:

- Calculate and Plot the x -intercepts (after factoring, if necessary): For this, set the function to 0, and solve for x . In case of only Imaginary Roots, there are NO x -intercepts.
- Calculate and Plot the y -intercept: For this, substitute $x = 0$, and simplify.
- Determine the End Behavior of the function: **See #13 above!** Draw arrows to indicate the end-behavior of the graph.
- Determine the behavior of the graph at each x -intercept, whether it “touches” or “crosses” the axis [see #15 above]. In general, if the x -intercept at r_1 corresponds to an odd-powered term in $f(x)$ [ie. $f(x) = \dots(x - r_1)^n \dots$ where n is odd], then $f(x)$ *crosses* the x -axis at $x = r_1$ (like a line cutting the axis). But if the x -intercept at r_2 corresponds to an even-powered term in $f(x)$ [ie. $f(x) = \dots(x - r_1)^n \dots$ where n is even], then $f(x)$ *touches* the x -axis at $x = r_2$ (like a parabola touching the axis).
- For an additional check of how the function behaves on different intervals along the x -axis as suggested by the x -intercepts [is $f(x) > 0$ or < 0 ? ie. above/below the axis?], a sign-table may be constructed using “test-values” for each interval.
- Draw a “logical” smooth curve taking a) – e) into account.

8: Rational Functions

1. A **Rational Function** is one of the form $f(x) = p(x)/q(x)$ where $p(x)$ and $q(x)$ are polynomial functions, $q(x) \neq 0$.

2 An **Asymptote** is a line that the graph of a function only approaches but never touches.

3 a) The line $x = a$ is called the **Vertical Asymptote** (V.A.) of a Rational Function, $f(x)$, if: as $x \rightarrow a^+$, $y \rightarrow +\infty$ or $-\infty$, and similarly, as $x \rightarrow a^-$, $y \rightarrow +\infty$ or $-\infty$.

b) To find the Vertical Asymptote of a rational function, $f(x) = p(x)/q(x)$, set the denominator, $q(x) = 0$ and solve for x .

4 a) The line $y = b$ is called the **Horizontal Asymptote** (H.A.) of a Rational Function, $f(x)$, if: as $x \rightarrow +\infty$, $y \rightarrow b$ or as $x \rightarrow -\infty$, $y \rightarrow b$. Loosely, the H.A. is the end behavior of the rational function.

b) To find the Horizontal Asymptote of a rational function, $f(x) = p(x)/q(x)$, consider *only* the leading terms [**See #1 above!**] of the numerator and denominator [ie. drop all terms of lower powers], simplify the expression in x , then take Limits as $x \rightarrow \pm\infty$, keeping in mind: If $a > 1$, then $a^\infty = \infty$; If $0 < a < 1$, then $a^\infty = \text{ZERO}$.

5. Sketching Rational Functions: $f(x) = p(x)/q(x)$:

- a) Calculate and Plot the x -intercepts (after factoring, if necessary): For this, set the numerator, $p(x) = 0$, and solve for x . In case of only Constants or Imaginary Roots for the numerator, there are NO x -intercepts.
- b) Calculate and Plot the y -intercept: For this, substitute $x = 0$, and simplify.
- c) Find and Plot the Vertical Asymptotes [using dotted lines].
- d) Find and Plot the Horizontal Asymptote [using dotted lines at the right and left “ends” of the graph since it is akin to the End Behavior of the function, $f(x)$].
- e) For an additional check of how the function behaves on different intervals along the x -axis as suggested by the x -intercepts and the Vertical Asymptotes [ie. is $f(x) > 0$ or < 0 ? ie. above/below the axis?], a sign-table may be constructed using “test-values” for each interval.
- f) Draw a “logical” smooth curve taking a) – e) into account.

9: Conic Sections

1. Identification of conic sections, $f(x) = 0$:

- a) Line: if $f(x)$ is linear in x and y
- b) Circle: if $f(x)$ is quadratic in x and y with the coefficients of x^2 and y^2 identical
- c) Parabola: if $f(x)$ is quadratic in x or y but *not*
- d) Ellipse: if $f(x)$ is quadratic in x and y with the coefficients of x^2 and y^2 *different* but having same *signs*
- e) Hyperbola: if $f(x)$ is quadratic in x and y with the signs of the coefficients of x^2 and y^2 being different

2. Definitions:

- a) Circle: the locus of points equidistant from a fixed point (the centre).
- b) Parabola: the locus of points equidistant from a fixed point (the vertex) as well as a line (the directrix).
- c) Ellipse: the locus of points such that the *sum* of distances from 2 fixed points (the Foci) is constant.
- d) Hyperbola: the locus of points such that the *difference* of distances from 2 fixed points (the Foci) is constant.

3 a) Standard Equation of a **Circle**: $x^2 + y^2 = r^2$ where $r \sim$ radius; the Centre is the Origin, $O(0, 0)$

b) Writing the Equation of a circle with centre, $C(h, k)$ and radius, r : *shift* the x and y terms of the standard equation of the circle, $x^2 + y^2 = r^2$, by h and k , respectively to get

$$(x - h)^2 + (y - k)^2 = r^2$$

c) Identifying a circle in form $ax^2 + ay^2 + bx + cy + d = 0$: first complete the square for the x and y terms to rewrite it in standard form, $(x - h)^2 + (y - k)^2 = r^2$. Then, the Centre is $C(h, k)$ with radius, r

d) Graphing a circle with center, $C(h, k)$: locate the center, C , then go up, down, right and left r units and plot 4 points! Join to form a smooth circle.

4 a) Standard Equation of a **Parabola**: $y^2 = 4ax$ or $x^2 = 4ay$ where a is the distance from the Vertex to the Focus or from the Vertex to the Directrix; the Vertex is the Origin, $O(0, 0)$

b) For $y^2 = 4ax$, the parabola goes left ($a < 0$) or right ($a > 0$). For $x^2 = 4ay$, the parabola goes down ($a < 0$) or up ($a > 0$). [See **NOTES #2: Family of Functions** o) and p)]

c) Writing the Equation of a parabola with Vertex at $V(h, k)$: *shift* the x and y terms of the standard equation of the parabola, $y^2 = 4ax$ or $x^2 = 4ay$ by h and k , respectively to get $(y - k)^2 = 4a(x - h)$ or $(x - h)^2 = 4a(y - k)$

d) Identifying a parabola in form $ax^2 + bx + cy + d = 0$ or $ay^2 + bx + cy + d = 0$: first complete the square for the quadratic term [x or y] taking the linear and constant terms on the other side of the equality, to rewrite it in standard form, $(y - k)^2 = 4a(x - h)$ or $(x - h)^2 = 4a(y - k)$. Then, the Vertex is $V(h, k)$, a units from the Focus and Directrix.

e) Graphing a parabola with vertex, $V(h, k)$: Plot Vertex first. *Then* parabola in the form for $y^2 = 4ax$ move a units right ($a > 0$) or left ($a < 0$) from the vertex to locate Focus and a units left ($a > 0$) or right ($a < 0$) from vertex to locate directrix line: $x = -a$. For parabola in the form or $x^2 = 4ay$ move a units up ($a > 0$) or down ($a < 0$) from the vertex to locate Focus and a units down ($a > 0$) or up ($a < 0$) from vertex to locate directrix line: $y = -a$.

5 a) Standard Equation of an **Ellipse**: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$;

where a is the distance from the center to the vertices lying on the horizontal axes (right and left) and b is the distance from the center to the vertices lying on the vertical axes (up and down).

The center is the Origin, $O(0, 0)$ and the 2 Foci lie along the *major* axis [horizontal if $a > b$; vertical if $a < b$].

Tip! While textbooks make a big deal of a and b [switching them around at times] it *doesn't* matter what symbols one uses: simply observe that the number below the x^2 term applies to vertices on the horizontal axis while that below the y^2 term applies to the vertices on the vertical axis! [See **NOTES #2: Family of Functions** q)]

b) c is the distance between the center and the 2 Foci, and is obtained by: $c^2 = l^2 - s^2$, where l [\sim long] and s [\sim short] represent a or b , whichever is longer or shorter! [see **Tip!** Above.] The Foci *always* lie along the longer axis.

c) Writing the Equation of an Ellipse with Centre at $C(h, k)$: *shift* the x and y terms of the standard equation of the ellipse, $x^2/a^2 + y^2/b^2 = 1$, by h and k , respectively to get: $(x - h)^2/a^2 + (y - k)^2/b^2 = 1$

d) Identifying an ellipse in form $ax^2 + by^2 + cx + dy + e = 0$: first complete the square for both quadratic terms [x^2 and y^2] taking the constant term to the other side of the equality, to rewrite it in standard form: $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$

Then, the Centre is $C(h, k)$, and a and b are as defined in 5a) above!

e) Graphing an ellipse with Centre, $C(h, k)$: Plot the Centre first. *Then* from the center move a units right and left to locate the 2 vertices on the horizontal axis; and move b units up and down from the center to locate the 2 vertices on the vertical axis. Depending on whether the ellipse has a horizontal axis [$a > b$] or vertical axis [$a < b$], plot the 2 Foci along the *major* axis using 5b).

6 a) Standard Equation of a **Hyperbola**:

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if the vertices lie on a horizontal axis [x^2 term is positive] **Case I**

$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ if the vertices lie on a vertical axis [y^2 term is positive] **Case II**

where a is the distance from the center to the vertices. Strictly speaking, *unlike the ellipse*, the hyperbola has only 1 pair of vertices (that it passes through) as dictated by a . The b^2 term [and consequently, b] term determines how *wide* the hyperbola opens and dictates the co-vertices [See 6 e) below].

The center is the Origin, $O(0, 0)$ and the 2 Foci lie *always* along the *major* axis.

Tip! While textbooks make a big deal of a and b [switching them around at times] it *doesn't* matter what *symbols* one uses: simply observe that the number below the x^2 term applies to the horizontal axis while that below the y^2 term applies to the vertical axis!

b) c is the distance between the center and the 2 Foci, and is obtained by: $c^2 = a^2 + b^2$. The Foci *always* lie along the *major* axis.

c) Writing the Equation of a Hyperbola with Centre at $C(h, k)$: *shift* the x and y terms of the standard equation of the hyperbola, $x^2/a^2 - y^2/b^2 = 1$ or $y^2/a^2 - x^2/b^2 = 1$, by h and k , respectively to get:

$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ **Case III** or $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ **Case IV**

d) Identifying a hyperbola in form $ax^2 - by^2 + cx + dy + e = 0$ or $ay^2 - bx^2 + cx + dy + e = 0$: first complete the square for both quadratic terms [x^2 and y^2] taking the constant term to the other side of the equality, to rewrite it in standard form **CASE III** or **CASE IV** above.

Then, the Centre is $C(h, k)$, and a and b are as defined in 6 a) above and 6 e) below!

e) Graphing a Hyperbola with Centre, $C(h, k)$: Plot the Centre first.

- For **CASE III**, from the center move a units right and left to locate the 2 vertices [See 6 a)] on the horizontal axis; and move b units up and down from the center to locate the 2 co-vertices on the vertical axis. Draw a rectangle with the vertices and co-vertices constituting the *mid-points*. Draw the diagonals of the rectangle and extend them beyond. Finally, draw a smooth hyperbola along the *horizontal* axis [right-left] passing through the vertices and with the diagonals of the rectangle as its asymptotes! Plot the 2 Foci along the *major* axis using 5b) at a distance of c from the centre.

- For **CASE IV**, from the center move a units up and down to locate the 2 vertices [See 6 a)] on the vertical axis; and move b units left and right from the center to locate the 2 co-vertices on the horizontal axis. Draw a rectangle with the vertices and co-vertices constituting the *mid-points*. Draw the diagonals of the rectangle and extend them beyond. Finally, draw a smooth hyperbola along the *vertical* axis [up-down] passing through the vertices and with the diagonals of the rectangle as its asymptotes! Plot the 2 Foci along the *major* axis using 5b) at a distance of c from the centre.

f) Asymptotes of the Hyperbola:

- For **CASE III**, the slope of the asymptote is, $m_1 = \pm b/a$ [from the center (h, k) , “rise” b units and “run” a units to get the top right vertex of the rectangle]. Use the slope-intercept form $[y = mx + b]$ or point slope form $[y - y_1 = m(x - x_1)]$, to find the equation of the asymptote!
- For **CASE IV**, the slope of the asymptote is, $m_1 = \pm a/b$ [from the center (h, k) , “rise” a units and “run” b units to get the top right vertex of the rectangle]. Use the slope-intercept form $[y = mx + b]$ or point slope form $[y - y_1 = m(x - x_1)]$, to find the equation of the asymptote!

10: Complex Numbers

1. A **Complex number** is the *non*-real solution to an equation of the form $(ax + b)^{p/q} = c$, where $c < 0$ and p is even. The square-root of a negative number produces the imaginary number, i , where $i = \sqrt{-1}$ so that $i^2 = -1$.

2. A complex number in **standard form** is a number in the form $a + bi$ where a and b are real numbers and i is the imaginary unit. If $b \neq 0$ then $a + bi$ is also called an imaginary number. If $a = 0$ then the complex number, bi , is called a pure imaginary number. If $b = 0$ then, the complex number, a , is a real number. The complex is zero if $0 + 0i = 0$.

3. The **Conjugate** of $a + bi$ is $a - bi$. That is, only the sign of the imaginary part is switched.

4. **Operations on Complex Numbers:** Because complex numbers obey the same commutative, associative, and distributive properties as real numbers, most operations with complex numbers are performed by using these properties and the fact that $i^2 = -1$.

a) **Equality:** Two complex numbers are equal if their real and imaginary components are equal: $a + bi = c + di$ if and only if $a = c$ and $b = d$.

b) **Addition:** To add / subtract 2 complex numbers, simply add/subtract their corresponding real and imaginary components: $(a + bi) \pm (c + di) = (a \pm c) + (b \pm di)$

c) **Multiplication:** For the product of 2 complex numbers, use FOIL to multiply the terms treating the “ i ” as a variable [like x], combine like terms [ie. the real and imaginary parts] and simplify using $i^2 = -1$.

d) **Quotient & Reciprocals:** For the quotient of 2 complex numbers $(a + bi) / (c + di)$ as well as to find the reciprocal of a complex number $1/(a + bi)$, multiply the numerator and denominator of the expression by the *conjugate* of the denominator. Then simplify by #4c) above, and using the property of conjugates, $(a + bi)(a - bi) = a^2 + b^2$.

11: Matrices and Determinants

1. A **Matrix** is a square / rectangular table of numbers with rows running horizontally, and columns aligned vertically. If a matrix, **A**, has m rows and n columns, then the *order* of the matrix is $m \times n$, also denoted $\mathbf{A}_{m \times n}$. A matrix may be rectangular [$m \neq n$] or square [$m = n$].

2. Two matrices are equal if their orders are identical, and the corresponding elements of the 2 matrices are the same.

3. Two matrices of the same order can be added / subtracted: in either case, corresponding elements of both matrices are simply added / subtracted.

4. When a matrix, A , is multiplied by a constant, k , then *every* element of A is multiplied by k .

5. To multiply 2 matrices, $A_{m \times n}$ and $B_{p \times q}$,

a) Number of columns of A must be equal to the number of rows of B ie. $n = p$

b) The order of the resultant matrix, C , shall be $m \times q$.

c) c_{ij} , the element in the i^{th} row and j^{th} column of matrix C , is obtained by the sum of the product of corresponding elements of the i^{th} row of A and the j^{th} column of B . For instance, to get c_{32} multiply and add the corresponding elements of row 3 of matrix A and column 2 of matrix B .

6. Properties of Matrix Multiplication: If A , B and C are matrices for which the following products are defined. Then,

a) Associative Property: $A(BC) = (AB)C$

b) Distributive Property: $A(B + C) = AB + AC$ and $(B + C)A = BA + CA$

c) Matrix Multiplication is *not* Commutative: $AB \neq BA$.

7. The **Identity Matrix**, I_n , is a square matrix of n rows and n columns such that all the elements along the main diagonal ie. a_{11} , a_{22} , etc. are 1s, and all the other elements are 0s.

8. The **Inverse** of a *square* Matrix, A of order n , is denoted as A^{-1} , and is a matrix that when pre-multiplied or post-multiplied with A yields the Identity Matrix of order n . That is, $A^{-1} \cdot A$ or $A \cdot A^{-1} = I$.

9. **Elementary Row Transformations:**

a) Switching 2 rows of a matrix

b) Adding / Subtracting 2 rows of a matrix and writing the result in one of the rows

c) Multiplying / Dividing any row of a matrix by a constant

d) Performing c) followed by b) in one step

10. The Inverse of a matrix, A , of order n , is obtained by constructing an Augmented Matrix, $[A_{n \times n} : I_n]$ and then performing elementary row transformations repeatedly on A , *and* identical operations simultaneously on I , to transform $[A_{n \times n} : I_n]$ into $[I_n : B_{n \times n}]$.

Then, $B = A^{-1}$.

11. A matrix is in **reduced echelon form** if

a) the 1st non-zero number, called the **leading entry**, in each row is 1.

b) the leading entry in each row lies to the right of the leading entry of the row immediately above it.

c) all rows consisting entirely of zeros lie at the bottom of the matrix.

d) every number above and below each leading entry is a zero.

12. Solving a System of Equations using Matrices: $AX = B$ where $A_{n \times n}$ is a square coefficient matrix, $X_{n \times 1}$ is the variable matrix and $B_{n \times 1}$ is the constant matrix.

a) Method I: $X = A^{-1}B$ by finding A^{-1} first through #9 above!

b) Method II: by constructing an Augmented Matrix $[A_{n \times n} : B_{n \times 1}]$ and then performing elementary row transformations repeatedly on A , *and* identical operations simultaneously on B to transform $[A_{n \times n} : B_{n \times 1}]$ into $[I_n : C_{n \times 1}]$.

c) Method III: Using Method II, it may suffice to construct an Augmented Matrix $[A_{n \times n} : B_{n \times 1}]$ and then performing elementary row transformations repeatedly on A , *and* identical operations simultaneously on B to transform $[A_{n \times n} : B_{n \times 1}]$ into $[D_{n \times n} : C_{n \times 1}]$ where $D_{n \times n}$ is matrix A is reduced echelon form [see #10 above!]. Then, by simply equating each row of $D_{n \times n}$ to the corresponding row of $C_{n \times 1}$, the variable of $X_{n \times 1}$ can be determined.

13. A Determinant is a square table of numbers with a single numerical value (upon simplification!). If $[a \ b]$ and $[c \ d]$ are the 2 rows of a 2×2 determinant, A , then its value is, $|A| = ad - bc$.

14a) The **Minor**, M_{ij} , of any element a_{ij} in determinant $A_{n \times n}$ is a determinant of order $(n-1 \times n-1)$ obtained by deleting the i -th row and j -th column of A .

b) The **Cofactor**, B_{ij} , of any element a_{ij} in determinant $A_{n \times n}$ is $|B_{ij}| = (-1)^{i+j}M_{ij}$

15. The value of the determinant $A_{n \times n}$, $|A|$, is the sum of the product of each element in *any* row or column and its co-factor.

16. Determinant Properties:

a) A square matrix, A , has an Inverse only if its determinant, $|A|$, is non-zero.

b) If any 2 rows or columns of a determinant are switched, the new value is *negative* (ie. -1 times) of the old.

c) A determinant obtained by adding a multiple of any row/column to another row/column *or* subtracting a multiple of any row to another row / column from another row / column does *not* change the value of the original determinant.

d) The value of a determinant with 2 identical rows / columns is *zero*.

17. Cramer's Rule for Solving Equations: If a system of n linear equations in n variables $x_1, x_2, x_3 \dots x_n$ is written in matrix notation as: $DX = B$ where $D_{n \times n}$ is the coefficient matrix, $X_{n \times 1}$ is the variable matrix and $B_{n \times 1}$ is constant matrix [see #11 above], then, if $|D| \neq 0$, the solutions are:

$$x_1 = |DX_1| / |D|, x_2 = |DX_2| / |D|, \dots x_n = |DX_n| / |D|$$

where Dx_i is the matrix obtained by replacing the i -th column of D by the column matrix, B .

12: Vectors

1. A vector $\mathbf{u} = a\mathbf{i} + b\mathbf{j} \sim \langle a, b \rangle$ possesses both magnitude and direction. a and b are the horizontal and vertical components of the vector, respectively, and represent the point (a, b) on the coordinate plane. The vector itself is an "arrow" from the origin $(0,0)$ to (a,b) .

2. The **magnitude** of a vector, denoted as $|\mathbf{u}|$, is its length: $|\mathbf{u}| = \sqrt{a^2 + b^2}$

The **direction** of a vector $\mathbf{u} = a\mathbf{i} + b\mathbf{j} \sim \langle a, b \rangle$ is the angle θ it makes with the positive x -axis. In general,
 $a = |\mathbf{u}| \cos \theta$ and $b = |\mathbf{u}| \sin \theta$ so that $\tan \theta = b/a \Rightarrow \theta = \tan^{-1}(b/a)$

Tip!

Based on the point (a, b) for $\mathbf{u} = a\mathbf{i} + b\mathbf{j} \sim \langle a, b \rangle$, identify the correct *quadrant* for the vector and "adjust" the angle θ , since $\theta = \tan^{-1}(b/a)$ gives the **reference angle** ie. the *smallest* angle the vector makes with the x -axis. The angle *needed* is the "total" angle that the vector makes with the *positive* x -axis!

3. A **unit vector** is a vector in the same direction as the "original" vector but with a magnitude of *one* (1). The unit vector associated with $\mathbf{u} = a\mathbf{i} + b\mathbf{j} \sim \langle a, b \rangle$, is $\mathbf{v} = a/|\mathbf{u}| \mathbf{i} + b/|\mathbf{u}| \mathbf{j}$ where $|\mathbf{u}|$ is the magnitude of \mathbf{u} . (See 2. above.) Confirm that the magnitude of such a vector is ALWAYS 1!

4. The **dot product** of 2 vectors $\mathbf{u} = a_1\mathbf{i} + b_1\mathbf{j}$ and $\mathbf{v} = a_2\mathbf{i} + b_2\mathbf{j}$, denoted by $\mathbf{u} \cdot \mathbf{v} = a_1 \cdot a_2 + b_1 \cdot b_2$

5. The angle, θ , between 2 vectors $\mathbf{u} = a_1\mathbf{i} + b_1\mathbf{j}$ and $\mathbf{v} = a_2\mathbf{i} + b_2\mathbf{j}$ is given by:
 $\cos \theta = \mathbf{u} \cdot \mathbf{v} / |\mathbf{u}| \cdot |\mathbf{v}| = (a_1 \cdot a_2 + b_1 \cdot b_2) / |\mathbf{u}| \cdot |\mathbf{v}|$
 In other words, the cosine of the angle, θ , is the ratio of the dot product of the 2 vectors and the product of their magnitudes.

6. 2 vectors are **orthogonal** - a big word for *perpendicular*! - if the angle between them is 90° . From #5 above, $\cos 90^\circ = 0$
 \Rightarrow If 2 vectors are orthogonal, $a_1 \cdot a_2 + b_1 \cdot b_2 = 0$

7. 2 vectors are **parallel** if the angle between them is 0° or 180° .
 Since, $\cos 0^\circ = 1$ and $\cos 180^\circ = -1$, using #5 above, if 2 vectors are parallel,
either numerator = denominator: $a_1 \cdot a_2 + b_1 \cdot b_2 = |\mathbf{u}| \cdot |\mathbf{v}|$, or
numerator = - denominator: $a_1 \cdot a_2 + b_1 \cdot b_2 = -(|\mathbf{u}| \cdot |\mathbf{v}|)$

13: Sequence, Series, The Binomial Theorem & Mathematical Induction

1. An **Arithmetic Sequence** is a sequence of the form $a, a + d, a + 2d, a + 3d, \dots$ where a is the **1st term** [$a_1 = a$], $a + d$ is the **2nd term** [$a_2 = a + d$], $a + 2d$ is the **3rd term** [$a_3 = a + 2d$], and so on, and d is the **common difference** between any term and the preceding one.

2 a) The ***n*th term** of an arithmetic sequence is: $a_n = a_1 + (n - 1)d$.

b) The sum of n terms of an arithmetic sequence is given by:

$$S_n = \frac{1}{2} n [a_1 + a_n] \text{ or } S_n = \frac{1}{2} n [2a_1 + (n - 1)d]$$

3. A **Geometric Sequence** is a sequence of the form a, ar, ar^2, ar^3, \dots where a is the **1st term** [$a_1 = a$], ar is the **2nd term** [$a_2 = ar$], ar^2 is the **3rd term** [$a_3 = ar^2$], and so on, and r is the **common ratio** between any term and the preceding one.

4 a) The n th term of a geometric sequence is: $a_n = a_1 r^{n-1}$

b) The sum of n terms of a geometric sequence is given by:

$$S_n = \frac{a_1(1 - r^n - 1)}{(1 - r)}$$

c) The sum of *infinite* terms of a geometric sequence with $-1 < r < 1$, is given by: $S_n = \frac{a}{(1 - r)}$

5 a) **Binomial Theorem:** $(x + y)^n = {}^n C_0 x^n y^0 + {}^n C_1 x^{n-1} y + {}^n C_2 x^{n-2} y^2 + {}^n C_3 x^{n-3} y^3 + \dots + {}^n C_n x^0 y^n$

b) The expansion of $(x + y)^n$ has $(n + 1)$ terms and the $(r + 1)^{\text{th}}$ term is given by ${}^n C_r x^{n-r} y^r$.

6. Principle of Mathematical Induction:

Step I: Demonstrate (through substitution) that the hypothesis [ie. The statement] holds true for $n = 1$ and $n = 2$.

Step II: *Assume* that the hypothesis holds true for some arbitrary $n = k$. [Simply substitute $n = k$ in the hypothesis.]

Step III: Prove that the hypothesis holds true for $n = k + 1$ [using Step II: since *part* of the Left Hand Side (LHS) of Step III is identical to the LHS of Step II, substitute the Right Hand Side (RHS) of Step II into the LHS of Step III]